Test 2 / Numerical Mathematics 1, May 31st 2022, University of Groningen

Instructions

- You have 2 hours to complete the test. When applicable, people with special facilities have 2h20 minutes in total.
- The exam is "closed book", meaning that you can only make use of the material given to you.
- The grade will be computed as the number of obtained points, plus 1.
- All answers need to be justified using mathematical arguments.

Exercise 1 (6 points)

Consider the function $f(x) = x^3 - 7x + 2$ with $x \in [0, 1]$.

(a) [0.5] Show that any fixed-point of $g(x) = (x^3 + 2)/7$ is also a root of $f(x)$.

Trivial

(b) [0.5] Show that if $x \in [0, 1]$, then $g(x) \in [0, 1]$.

Trivial

(c) [1.0] First show that $g(x)$ is a contraction on $x \in [0, 1]$.

Derivate **(0.5 pt)**, then take the max **(0.25 pt)**. Then show that $L < 1$ **(0.25 pt)**.

(d) [0.5] Show that $g(x)$ has a unique fixed-point on $x \in [0, 1]$.

Existence is assured by b **(0.1 pt)**.
Assume two fixed points **(0.2 pt)**. Then using that $g(x)$ is a contraction, one gets unicity **(0.2 pt)**.

(e) [2.0] Show that the sequence $x_{k+1} = g(x_k)$, $k \geq 0$ converges to the root of $f(x)$ in $x \in [0, 1]$ if $x_0 \in [0, 1]$.

$|x_{k+1} - x^*| = |g(x_k) - g(x^*)|$ **(0.5 pt)**, $\leq L|x_k - x^*|$ **(0.5 pt)**, $\leq L^{k+1}|x_0 - x^*|$ **(0.5 pt)**. Then it goes to zero when k goes to ∞ **(0.5 pt)**.

(f) [0.5] Write the general formula for the Newton iteration for the problem $f(x) = 0$, i.e. give an explicit form for computing x_{k+1} in terms of x_k.

$$x_{k+1} = x_k - \frac{x_k^3 - 7x_k + 2}{3x_k^2 - 7} \textbf{(0.5 pt)}$$

No points for stating the formula for a general function $f(x)$.

(g) [1.0] Starting from $x_0 = 1$, perform: i) one fixed-point iteration with $g(x)$, and ii) one Newton iteration. Which method leads to a more accurate approximation $x_1 \approx x_*$ if the root is $x_* \approx 0.289$?

i) $x_1 = (1^3 + 2)/7 = 3/7$ **(0.4 pt)**
ii) $x_1 = 1 - \frac{1^3 - 7 \cdot 1 + 2}{3 \cdot 1^2 - 7} = 1 - \frac{1 - 7 + 2}{3 - 7} = 0.$ **(0.4 pt)**
The Newton method seems to give a larger error **(0.2 pt)**. Note that we could ask about given reasons for this but they can be multiple.

Exercise 2 (3 points)

We want to solve the linear system $Ax = b$ for $x \in \mathbb{R}^2$ by using stationary Richardson iterations:

$$x^k = x^{k-1} + \alpha \left(b - Ax^{k-1}\right)$$

using as initial guess the vector $x^0 = [1, 0]^\intercal$, $b = [0, \gamma]^\intercal$. The matrix A is given by:

$$A = \begin{bmatrix} g & -d \\ -d & g \end{bmatrix} \;, \; g > d > 0.$$

(a) 1.0 Compute x^1 from x^0 in terms of α, g, d, γ.

$$x^1 = [1 \; 0]^\intercal + \alpha\left([0 \; 1]^\intercal - A[1 \; 0]^\intercal\right) = [1 - \alpha g \;, \; 0 + \alpha\gamma + d\alpha]^\intercal$$

(0.5 pt) for each vector component.

(b) 2.0 Give a value of α in terms of g, γ and d so that convergence of the Richardson iterations towards $A^{-1}b$ is ensured. Justify your answers in view of the theory.

The eigenvalues of A are

$$\lambda_\pm = \frac{2g \pm \sqrt{4g^2 - 4(g^2 - d^2)}}{2} = g \pm d > 0$$

(**(0.5 pt)** for each correct eigenvalue). We need to choose α such that the spectral radius of the iteration matrix $I - \alpha A$ for the error is smaller than 1 **(0.5 pt)**. Since A is SPD, any positive value smaller than $2/\lambda_+ = 2/(g+d)$ will ensure convergence **(0.5 pt)**.